LOGARITHMIC SERIES

* $\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots \infty$

* $\log(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} + \dots \infty$

$\Rightarrow \log(1-x) = -\left(x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \dots \infty\right)$

* $\log\left(\frac{1+x}{1-x}\right) = 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \dots \infty\right)$

Note:

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^3}{4} + \dots \infty$$

$x = 1$

$$\log 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots \infty$$

Ex 1 Prove that

$$\log \frac{n+1}{n-1} = \frac{2n}{n^2+1} + \frac{1}{3}\left(\frac{2n}{n^2+1}\right)^3 + \frac{1}{5}\left(\frac{2n}{n^2+1}\right)^5 + \dots \infty$$

Solution:

$RHS = x + \frac{x^3}{3} + \frac{x^5}{5} + \dots \infty$ where $x = \frac{2n}{n^2+1}$

$$RHS = \frac{1}{2}\log\left(\frac{1+x}{1-x}\right)$$

$$= \frac{1}{2}\log\left(\frac{1+\frac{2n}{n^2+1}}{1-\frac{2n}{n^2+1}}\right)$$

$$= \frac{1}{2}\log\left(\frac{n^2+1+2n}{n^2+1-2n}\right)$$

$$= \frac{1}{2}\log\left(\frac{n+1}{n-1}\right)^2 = \log\left(\frac{n+1}{n-1}\right) = RHS$$

Ex-2. Show that $\log(x+2h) = 2\log(x+h) - \log x - \left[\frac{h^2}{(x+h)^2} + \frac{h^4}{2(x+h)^4} + \frac{h^6}{3(x+h)^6} + \cdots \infty\right]$

Solution:

$$RHS = 2\log(x+h) - \log x - \left[y + \frac{y^2}{2} + \frac{y^3}{3} + \cdots\right]$$

where $y = \frac{h^2}{(x+h)^2}$

$$= 2\log(x+h) - \log x + \log(1-y)$$

$$= 2\log(x+h) - \log x + \log\left[1 - \frac{h^2}{(x+h)^2}\right]$$

$$= 2\log(x+h) - \log x + \log\left[\frac{(x+h)^2 - h^2}{(x+h)^2}\right]$$

$$= 2\log(x+h) - \log x + \log\left[\frac{x(x+2h)}{(x+h)^2}\right]$$

$$= 2\log(x+h) - \log x + \log x + \log(x+2h) - 2\log(x+h)$$

$$= \log(x+2h).$$

Ex-3. Show that:

$$\log_e\left(1+\frac{1}{n}\right)^n = 1 - \frac{1}{2(n+1)} - \frac{1}{2\cdot 3(n+1)^2} - \frac{1}{3\cdot 4(n+1)^3} - \cdots \infty$$

Solution:

$$RHS = 1 - \frac{1}{2(n+1)} - \frac{1}{2\cdot 3(n+1)^2} - \frac{1}{3\cdot 4(n+1)^3} - \cdots \infty$$

$$= 1 - \left(1-\frac{1}{2}\right)\frac{1}{n+1} - \left(\frac{1}{2}-\frac{1}{3}\right)\frac{1}{(n+1)^2} - \left(\frac{1}{3}-\frac{1}{4}\right)\frac{1}{(n+1)^3} - \cdots \infty$$

$$= \left[1 + \frac{1}{2(n+1)} + \frac{1}{3(n+1)^2} + \dots \infty\right]$$

$$- \left[\frac{1}{n+1} + \frac{1}{2(n+1)} + \frac{1}{3(n+1)^3} + \dots \infty\right]$$

$$= (n+1)\left[\frac{1}{n+1} + \frac{1}{2(n+1)^2} + \frac{1}{3(n+1)^3} + \dots \infty\right]$$

$$- \left[\frac{1}{n+1} + \frac{1}{2(n+1)^2} + \frac{1}{3(n+1)^3} + \dots \infty\right]$$

$$= (n+1-1)\left[\frac{1}{n+1} + \frac{1}{2(n+1)^2} + \frac{1}{3(n+1)^3} + \dots \infty\right]$$

$$= -n \log\left[1 - \frac{1}{n+1}\right]$$

$$= -n \log\left(\frac{n}{n+1}\right) = n \log\left(\frac{n+1}{n}\right)$$

$$= n \log\left(1 + 1/n\right)$$

$$= n\left(1 + 1/n\right)^n = LHS.$$

Ex 4 Show that

$$3/4\left[\log_e 10 + 1/27 + 1/2 \cdot 3/2^{14} + 1/3 \cdot 3^2/2^{21} + \dots \infty\right] = \log 2$$

Solution:

$$LHS = 1/10\left[3 \log 10\ 3/27 + 1/2 \cdot \left(3/27\right)^2 + 1/3\left(3/27\right)^3 + \dots \infty\right]$$

$$= 1/10\left[3 \log 10 - \log\left(1 - 3/27\right)\right]$$

$$= 1/10\left[3 \log 10 - \log \frac{125}{128}\right]$$

$$= 1/10 \log \frac{1000 \times 128}{125}$$

$$= 1/10 \log 8 \times 128$$

$$= 1/10 \log 2^{10} = \log 2 = RHS$$

Ex5 Sum to infinity the series :

$$\left(1+\frac{1}{2}\right)+\left(\frac{1}{3}+\frac{1}{4}\right)\cdot\frac{1}{9}+\left(\frac{1}{5}+\frac{1}{6}\right)\cdot\frac{1}{9^2}+\ldots$$

Solution:

$$\left(1+\frac{1}{2}\right)+\left(\frac{1}{3}+\frac{1}{4}\right)\cdot\frac{1}{9}+\left(\frac{1}{5}+\frac{1}{6}\right)\cdot\frac{1}{9^2}+\ldots\infty$$

$$=\left(1+\frac{1}{3}\cdot\frac{1}{9}+\frac{1}{5}\cdot\frac{1}{9^2}+\ldots\infty\right)+$$

$$\left(\frac{1}{2}+\frac{1}{4}\cdot\frac{1}{9}+\frac{1}{6}\cdot\frac{1}{9^2}+\ldots\infty\right)$$

$$=\left(1+\frac{1}{3}\cdot\frac{1}{3^2}+\frac{1}{5}\cdot\frac{1}{3^4}+\ldots\infty\right)$$

$$+\frac{1}{2}\left(1+\frac{1}{2\cdot 9}+\frac{1}{3\cdot 9^2}+\ldots\infty\right)$$

$$=3\left(\frac{1}{3}+\frac{1}{3}\cdot\frac{1}{3^3}+\frac{1}{5}\cdot\frac{1}{3^5}+\ldots\infty\right)$$

$$+\frac{9}{2}\left(\frac{1}{9}+\frac{1}{2\cdot 9^2}+\frac{1}{3\cdot 9^3}+\ldots\infty\right)$$

$$=\frac{3}{2}\log\left(\frac{1+\frac{1}{3}}{1-\frac{1}{3}}\right)-\frac{9}{2}\log\left(1-\frac{1}{9}\right)$$

$$=\frac{3}{2}\log 2-\frac{9}{2}\log\frac{8}{9}$$

$$=\frac{3}{2}\log 2-\frac{9}{2}(3\log 2-2\log 3)=9\log 3-12\log 2$$

Ex-6 Show that if $x>0$,

$$\log x=\frac{x-1}{x+1}+\frac{1}{2}\cdot\frac{x^2-1}{(x+1)^2}+\frac{1}{3}\cdot\frac{x^3-1}{(x+1)^3}+\ldots$$

Solution:

$$LHS=\frac{x}{x+1}+\frac{1}{2}\cdot\frac{x^2}{(x+1)^2}+\frac{1}{3}\cdot\frac{x^3}{(x+1)^3}+\ldots\infty$$

$$-\left[\frac{1}{x+1}+\frac{1}{2(x+1)^2}+\frac{1}{3(x+1)^3}+\ldots\infty\right]$$

$$= -\log\left[1-\frac{x}{x-1}\right]+\log\left(1-\frac{1}{x+1}\right)$$

$$= \log\frac{x}{x+1}(x+1) = \log x = \text{LHS}.$$

Ex 7. If a, b, c denote three consecutive integers show that,

$$\log_e b = \frac{1}{2}\log_e a + \frac{1}{2}\log_e c + \frac{1}{2ac+1} + \frac{1}{3(2ac+1)^3} + \frac{1}{5(2ac+1)^5} + \ldots\infty$$

Solution :

$$\text{RHS} = \frac{1}{2}\log a + \frac{1}{2}\log c + x + \frac{x^3}{3} + \frac{x^5}{5} + \ldots\infty$$

where $x = \frac{1}{2ac+1}$

$$= \frac{1}{2}\log ac + \frac{1}{2}\log\left(\frac{1+x}{1-x}\right)$$

$$= \frac{1}{2}\log ac + \frac{1}{2}\log\left(\frac{1+\frac{1}{2ac+1}}{1-\frac{1}{2ac+1}}\right)$$

$$= \frac{1}{2}\log ac + \frac{1}{2}\log\left(\frac{2ac+2}{2ac}\right)$$

$$= \frac{1}{2}\log ac + \frac{1}{2}\log\left(\frac{ac+1}{ac}\right)$$

$$= \frac{1}{2}\log ac + \frac{1}{2}\log(ac+1) - \frac{1}{2}\log ac$$

$$= \frac{1}{2}\log(ac+1)$$

Since :

a, b, c are 3 consecution integers.

$a = b - 1$

$\& \; c = b + 1$

$\therefore ac = b^2 - 1$

$ac + 1 = b^2$

$\therefore RHS = \frac{1}{2} \log b^2 = \log b = LHS.$

Ex-8 if α and β are the roots of $x^2 - px + q = 0$ Show that

$$\log(1 + px + qx^2) = (\alpha + \beta)x - (\alpha^2 + \beta^2)\frac{x^2}{2} + (\alpha^3 + \beta^3)\frac{x^3}{3} - \dots \infty$$

Solution :

$\alpha + \beta$ are the roots of $x^3 - px + q = 0$

$\therefore \alpha + \beta = p$

$\alpha\beta = q$

$$\log(1 + px + qx^2) = \log(1 + (\alpha+\beta)x + \alpha\beta x^2)$$

$$= \log(1 + ax)(1 + \beta x)$$

$$= \log(1 + ax) + \log(1 + \beta x)$$

$$= \alpha x - \frac{\alpha^2 x^2}{2} + \frac{\alpha^3 x^3}{3} - \dots \infty$$

$$= \beta x = \frac{\beta^2 x^2}{2} + \frac{\beta^3 x^3}{3} - \dots \infty$$

$$= (\alpha + \beta)x - (\alpha^2 + \beta^2)\frac{x^2}{2} + (\alpha^3 + \beta^3)\frac{x^3}{3} - \dots$$

Ex-9 if $x^2 y = 2x - y$ and $x < 1$ show that

$$y + \frac{y^3}{3} + \frac{y^5}{5} + \dots = 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \dots\right)$$

Solution :

$x^2 y = 2x - y$

$y(x^2 + 1) = 2x.$

$$y = \frac{2x}{x^2+1}$$

$$LHS = y + y^3/_3 + y^5/_5 + \dots \infty$$

$$= \frac{1}{2} \log \left(\frac{1+y}{1-y} \right)$$

$$= \frac{1}{2} \log \left(\frac{1 + \left(\frac{2x}{1+x^2}\right)}{1 - \frac{2x}{1+x^2}} \right) = \frac{1}{2} \log \left(\frac{1+x}{1-x} \right)^2$$

$$= \frac{1}{2} \qquad = \log \left(\frac{1+x}{1-x} \right)$$

$$= 2x + x^3/_3 + x^5/_5 + \dots \infty$$

Show that

Ex-10 $$1 + \left(\frac{1}{2} + \frac{1}{3}\right) \frac{1}{4} + \left(\frac{1}{4} + \frac{1}{5}\right) \frac{1}{4^2} + \left(\frac{1}{6} + \frac{1}{7}\right) \frac{1}{4^3}$$

$$+ \dots \infty = \log \sqrt{12}$$

Solution:

$$LHS = \left(\frac{1}{2} \cdot \frac{1}{4} + \frac{1}{4} \cdot \frac{1}{4^2} + \frac{1}{6} \cdot \frac{1}{4^3} + \dots \infty \right)$$

$$+ \left(1 + \frac{1}{3} \cdot \frac{1}{4} + \frac{1}{5} \cdot \frac{1}{4^2} + \dots \infty \right)$$

$$= \frac{1}{2} \left(\frac{1}{4} + \frac{1}{2} \cdot \frac{1}{4^2} + \frac{1}{3} \cdot \frac{1}{4^3} + \dots \infty \right)$$

$$+ 2 \left(\frac{1}{2} + \frac{1}{3} \cdot \frac{1}{2^3} + \frac{1}{5} \cdot \frac{1}{2^5} \right)$$

$$= -\frac{1}{2} \log \left(1 - \frac{1}{4} \right) + \log \left(\frac{1 + \frac{1}{2}}{1 - \frac{1}{2}} \right)$$

$$= \frac{1}{2} \log \frac{4}{3} + \log 3.$$

$$= \log \left(\sqrt{\frac{4}{3}} \cdot 3 \right) = \log \sqrt{12}.$$

Ex-11. if $y = x - x^2/2 + x^3/3 - x^4/4 + \dots \infty$ show that

$$x = y + \frac{y^2}{\lfloor 2} + \frac{y^2}{\lfloor 3} + \dots \infty$$

Solution:

$$y = x - \frac{x^2}{2} + \frac{x^3}{3} - x^4/4 + \dots \infty$$

i.e. $y \log_e (1+x)$

$\therefore 1+x = e^y$

$$1+x = 1 + y/\lfloor 1 + \frac{y^2}{\lfloor 2} + \frac{y^2}{\lfloor 3} + \dots \infty$$

Ex-12. Show that

$$\frac{1}{1.2.3} + \frac{1}{3.4.5} + \frac{1}{5.6.7} + \dots \infty = \log 2 - \frac{1}{2}$$

Solution:

The nth term of the series is

$$T_n = \frac{1}{(2n-1)(2n)(2n+1)}$$

let $$\frac{1}{(2n-1)(2n)(2n+1)} = \frac{A}{2n-1} + \frac{B}{2n} + \frac{C}{2n+1}$$

$$1 = A(2n)(2n+1) + B(2n-1)(2n+1) + C(2n-1)(2n)$$

put $n = 1/2$; $1 = A.1.2 \therefore A = 1/2$

put $n = 0$; $1 = -B \therefore B = -1$

put $n = -1/2$ $1 = C(-2)(-1) \therefore C = 1/2$

$$T_n = \frac{1/2}{2n-1} - \frac{1}{2n} + \frac{1/2}{2n+1}$$

put $n = 1, 2, 3 \dots$

$$T_1 = \frac{1/2}{1} - \frac{1}{2} + \frac{1/2}{3}$$

$$T_2 = \frac{1/2}{3} - \frac{1}{4} + \frac{1/2}{5}$$

$T_3 = \frac{1/2}{5} - \frac{1}{6} + \frac{1/2}{7}$

....

Adding $S_\infty = \frac{1}{2} + (-\frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots \infty$

$= \frac{1}{2} + (1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots \infty - 1)$

$= \frac{1}{2} + (\log 2 - 1)$

$= \log 2 - \frac{1}{2}$.

Ex - 13 Sum to infinity the series

$\frac{1}{1.1.3} + \frac{1}{2.3.5} + \frac{1}{3.5.7} + \dots \infty$

Solution :

the nth term of the series.

$T_n = \frac{1}{n(2n-1)(2n+1)} = \frac{2}{(2n-1)(2n)(2n+1)}$

let $\frac{2}{2n(2n-1)(2n+1)} = \frac{A}{2n-1} + \frac{B}{2n} + \frac{C}{2n+1}$

$2 = A\,2n(2n+1) + B(2n-1)(2n+1) + C(2n-1)2n$

put $n = \frac{1}{2}$; $A.1\ 2 = 2 \therefore A = 1$

put $n = 0$; $-B = 2 \therefore B = -2$

put $n = -\frac{1}{2}$; $C(-2)(-1) = 2 \therefore C = 1$

$\therefore T_n = \frac{1}{2n-1} - \frac{2}{2n} + \frac{1}{2n+1}$.

put $n = 1, 2, 3, \dots$

$T_1 = \frac{1}{1} - \frac{2}{2} + \frac{1}{3}$

$T_2 = \frac{1}{3} - \frac{2}{4} + \frac{1}{5}$

$T_3 = \frac{1}{5} - \frac{2}{6} + \frac{1}{7}$

$\ldots\ldots$

Adding $S = 1 + 2\left(-\frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots \infty\right)$

$= 1 + 2(\log 2 - 1)$

$= 2\log 2 - 1$

Ex-14. Sum to infinity the series $\frac{x}{1.2} + \frac{x^2}{2.3} + \frac{x^3}{3.4} + \ldots \infty$

Solution:

$T_n = \frac{x^n}{n(n+1)}$

$= \left(\frac{1}{n} - \frac{1}{n+1}\right) x^n$

$= \frac{x^n}{n} - \frac{x^n}{n+1}$

$= \frac{x^n}{n} - \frac{1}{x} \cdot \frac{x^{n+1}}{n+1}$

putting $n = 1, 2, 3, \ldots$ we get,

$T_1 = \frac{x}{1} - \frac{1}{x} \cdot \frac{x^2}{2}$

$T_2 = \frac{x^2}{2} - \frac{1}{x} \cdot \frac{x^3}{3}$

$T_3 = \frac{x^3}{3} - \frac{1}{x} \cdot \frac{x^4}{4}$

$\ldots\ldots$

Adding $S = \left(x + \frac{x^2}{2} + \frac{x^3}{3} + \ldots \infty\right) - \frac{1}{x}\left(\frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4}\right.$

$= -\log(1-x) - \frac{1}{x}\left(-\log(1-x) - x\right)$

$= -\log(1-x) + \frac{1}{x}\log(1-x) + 1$

$= \left(\frac{1}{x} - 1\right)\log(1-x) + 1.$

Ex-15. Sum to infinity the Series $\sum_{n=1}^{\infty} \frac{n^2}{(n+1)(n+2)} x^n$

Solution

let $\frac{n^2}{(n+1)(n+2)} = 1 \frac{A}{n+1} + \frac{B}{n+2}$

$\therefore n^2 = (n+1)(n+2) + A(n+2) + B(n+1)$

put $n = -1$; $A = 1$ or $A = 1$

put $n = -2$; $-B = 4$ $\therefore$ $B = -4$.

$$T_n = \left(1 + \frac{1}{n+1} - \frac{4}{n+2}\right) x^n$$

$$= x^n + \frac{x^n}{n+1} - \frac{4x^n}{n+2}$$

$$= x^n + \frac{1}{x}\frac{x^{n+1}}{n+1} - \frac{4}{x^2}\frac{x^{n+2}}{n+2}$$

$S = \sum T_n$

$$= \sum_{n=1}^{\infty} x^n + \frac{1}{x}\sum_{n=1}^{\infty}\frac{x^{n+1}}{n+1} - \frac{4}{x^2}\sum_{n=1}^{\infty}\frac{x^{n+2}}{n+2}$$

$$= (x + x^2 + x^3 + \dots \infty)$$

$$+ \frac{1}{x}\left(\frac{x^2}{2} + \frac{x^3}{3} + \dots \infty\right) - \frac{4}{x^2}\left(\frac{x^3}{3} + \frac{x^4}{4} + \frac{x^5}{5} + \dots\right)$$

$$= \frac{x}{1-x} + \frac{1}{x}\left[-\log(1-x) - x\right] - \frac{4}{x^2}\left[-\log(1-x) - x - \frac{x^2}{2}\right]$$

$$= \left(\frac{4}{x^2} - \frac{1}{x}\right)\log(1+x) + \frac{x}{1-x} + \frac{4}{x} + 1.$$

Ex-17. Prove that if n is very large

$\left(\frac{n+1}{n-1}\right)^n = e^2\left(1 + \frac{2}{3n^2}\right)$ approximately

Solution: $\left(\frac{n+1}{n-1}\right)^n = e^{n \log\left(\frac{n+1}{n-1}\right)}$

$$= e^{n \log\left(\frac{1+1/n}{1-1/n}\right)}$$

$$= e^{2n\left(1/n + \frac{1}{3n^3} + \frac{1}{5n^5} + \dots \infty\right)}$$

$$= e^{2 + \frac{2}{3n^2} + \frac{2}{5n^4} + \dots \infty}$$

$$= e^2 \left[1 + \frac{\frac{2}{3n^2} + \frac{2}{5n^4}}{\lfloor 1} + \dots \infty\right]$$

$$= e^2 \; 1 + \frac{2}{3n^2} \text{ approximately.}$$

Ex : 18, if n is very large prove that $\left(1 + 1/n\right)^n = e\left(1 - 1/2n + 11/24n^2\right)$ nearly.

Solution :

$$\left(1 + 1/n\right)^n = e^{n \log(1 + 1/n)}$$

$$= e^{n\left(1/n - 1/2n^2 + 1/3n^3 \dots \infty\right)}$$

$$= e^{1 - 1/2n + 1/3n^2 \dots \infty}$$

$$= e \cdot e^{\frac{-1}{2n} + \frac{1}{3n^2} + \dots \infty}$$

$$= e\left[1 + \frac{\left(-1/2n + 1/3n^2\right)}{\lfloor 1} + \frac{\left(-1/2n + 1/3n^2\right)^2}{\lfloor 2} + \dots \infty\right]$$

$$= e\left[1 - 1/2n + 1/3n^2 + 1/8n^2\right]$$

$$= e\left[1 - 1/2n + 11/24n^2\right]$$

if $\log(1-x+x^2) = a_1 x + a_2 x^2 + a_n x^3 + \dots \infty$

Show that,

$a_3 + a_6 + a_9 + \dots = \frac{3}{2} \log 2.$

Solution :

$$\log(1-x+x^2) = \log\left(\frac{1+x^3}{1+x}\right)$$

$$= \log(1+x^3) - \log(1+x).$$

$$= \left(x^3 - \frac{(x^3)^2}{2} + \frac{(x^3)^3}{3} - \dots \infty\right)$$

$$- \left(x - \frac{x^2}{2} + \frac{x^3}{3} \dots \infty\right)$$

$$= a_1 x + a_2 x^2 + a_3 x^3 + \dots \infty$$

Equating the coefficient of x^3, $a_3 = 1 - \frac{1}{3}$

Equating the coefficient of x^6, $a_6 = \frac{-1}{2} + \frac{1}{6}$

Equating the coefficient of x^9, $a_9 = \frac{1}{3} + \frac{1}{9}$

Adding $a_3 + a_6 + a_9 + \dots = \left(1 - \frac{1}{2} + \frac{1}{3} \dots \infty\right)$

$$-\frac{1}{3}\left(1 - \frac{1}{2} + \frac{1}{3} \dots \infty\right)$$

$$= \log 2 - \frac{1}{3} \log 2.$$

$$= \frac{2}{3} \log 2$$

Ex-20 Show that $\log\left(\frac{1+2e^x}{3}\right) = \frac{2x}{x} + \frac{x^3}{9}$

approximately.

Solution :

Show that $\log\left(\frac{1+2e^x}{3}\right) = \frac{2x}{3} + \frac{x^2}{9}$

Solution :

$$\log\left(\frac{1+2e^x}{3}\right) = \log\left[\frac{1+2\left(1+\frac{x}{1!}+\frac{x^2}{2!}+\cdots\right)}{3}\right]$$

$$= \log\left(1+\frac{2x}{3}+\frac{x^2}{3}+\cdots\right)$$

$$= \frac{2x}{3}+\frac{x^2}{3}-\frac{2x^2}{9}$$

$$= \frac{2x}{3}+\frac{x^2}{9} \text{ approx.}$$

Ex-21 when x is small show that.

$$\log\left[(1+x)^{1/2}+(1-x)^{1/3}\right] = \log 2 - x^2/9 \text{ nearly.}$$

Solution :

$$\log\left[(1+x)^{1/3}+(1+x)^{1/2}\right]$$

$$= \log\left[1+\tfrac{1}{3}x+\frac{\frac{1}{3}\left(-\frac{2}{3}\right)}{2!}x^2+\cdots\infty + 1-\tfrac{1}{3}x+\frac{\frac{1}{3}(-2/2)}{2!}x^2-\cdots\infty\right]$$

$$= \log\left[2-\frac{2x^2}{9}\right] \text{ nearly.}$$

$$= \log 2\left(1-x^2/9\right)$$

$$= \log 2+\log\left(1-x^2/9\right)$$

$$= \log 2 - x^2/9 - \frac{\left(x^2/9\right)^2}{2}+\cdots$$

$$= \log 2 - x^2/9 \text{ approx.}$$

Ex. 22 Expand $\log(1+x+x^2)$ in powers of x and show that the coefficient of x^n is either $-2/n$ or $1/n$ according as n is or not a multiple of 3.

Solution:

$$\log(1+x+x^2) = \log\left(\frac{1-x^3}{1-x}\right)$$

$$= \log(1-x^3) - \log(1-x).$$

$$= -\left[x^3 + \frac{(x^3)^2}{2} + \frac{(x^3)^3}{3} + \dots + \frac{(x^3)^r}{r} + \dots \infty\right]$$

$$+ \left[x + \frac{x^2}{2} + \frac{x^3}{3} + \dots + \frac{x^r}{r} + \dots \infty\right]$$

Suppose n is a multiple of 3. let $n = 3r$

the coefficient of x^n = coefficient of x^{3r},

$$= \frac{-1}{r} + \frac{1}{3r} = \frac{-3+1}{3r} = \frac{-2}{3r}.$$

$$= -2/n$$

Suppose n is not a multiple of 3. then the coefficient of x^n in the 1st expansion is zero

$\therefore$ the coefficient of x^n in the whole expansion $= 1/n$.